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On Existence and Uniqueness of Solutions of Boundary Value Problems of Fourth Order Elliptic Partial Differential Equations with Variable Coefficients

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The existence and uniqueness of solutions of a large class of boundary value problems of fourth order elliptic partial differential equations with variable/constant coefficients have been established. As examples of practical interest, bending problems of elastic anisotropic/orthotropic/isotropic plates with variable/constant thickness have been considered to illustrate the applicability of these results.

1. INTRODUCTION

In [2] sufficient conditions for the existence and uniqueness of solutions of the Dirichlet problem of fourth order elliptic partial differential equations with variable coefficients in weak form, which, in the case of bending problems of elastic anisotropic/orthotropic/isotropic plates with variable/constant thickness, corresponds to the clamped boundary conditions along the edge of the plate, are given. But these results of [2] can not be directly extended to other types of boundary conditions which are of great practical interest and importance. Hence, it is suggested that one proves the existence and uniqueness of solutions of these fourth order elliptic equations with variable coefficients in weak form for boundary conditions other than those of the Dirichlet problem in [2]. This paper contains new interesting results in this direction which are easily applicable.

2. NOTATIONS

Let Ω be a bounded open subset of $\mathbb{R}^2$ with boundary Γ which is assumed to be "sufficiently" smooth such that $\bar{\Omega} = \Omega \cup \Gamma$. Let $H^m(\Omega)$ be the usual Sobolev space [1, 3] of integral order $m \geqslant 0$ equipped with

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inner product $\langle \cdot, \cdot \rangle_{m,\Omega}$, norm $\|\cdot\|_{m,\Omega}$, and semi-norm $|\cdot|_{m,\Omega}$ such that $H^0(\Omega) \equiv L^2(\Omega)$,

$$H_0^1(\Omega) = \{v: v \in H^1(\Omega), \gamma_0 v = v|_\Gamma = 0\}, \tag{2.1}$$

$$H_0^2(\Omega) = \left\{v: v \in H^2(\Omega), \gamma_0 v = v|_\Gamma = 0, \gamma_1 v = \left(\frac{\partial v}{\partial n}\right)\bigg|_\Gamma = 0\right\}, \tag{2.2}$$

where γ_k: $H^m(\Omega) \to H^{m-k-1/2}(\Gamma)$ are trace operators, $k = 0, m-1$; $m = 1, 2$; $H^{3/2}(\Gamma)$, $H^{1/2}(\Gamma)$ being the fractional order Sobolev spaces on Γ [1, 3].

In addition to these spaces, the following closed subspaces of $H^m(\Omega)$, $m = 1, 2$, will be required: For nonempty open subsets $\Gamma_0, \Gamma_1 \subset \Gamma$ with $\text{meas}(\Gamma_i) \neq 0$ $(i = 0, 1)$

$$H_0^1(\Omega; \Gamma_0) = \{v: v \in H^1(\Omega), v|_{\Gamma_0} = 0\}, \tag{2.3}$$

$$H_0^2(\Omega; \Gamma_0, \Gamma_1) = \{v: v \in H^2(\Omega), v|_{\Gamma_0} = 0, (\partial v/\partial n)|_{\Gamma_1} = 0\}, \tag{2.4}$$

$$H^2(\Omega) \cap H_0^1(\Omega; \Gamma_0) = \{v: v \in H^2(\Omega), v|_{\Gamma_0} = 0\}. \tag{2.5}$$

In particular, for

$$\Gamma_0 = \Gamma, H_0^1(\Omega; \Gamma) = H_0^1(\Omega), H^2(\Omega) \cap H_0^1(\Omega; \Gamma) = H^2(\Omega) \cap H_0^1(\Omega); \tag{2.6}$$

for

$$\Gamma_0 = \Gamma_1, \qquad H_0^2(\Omega; \Gamma_0, \Gamma_0) = H_0^2(\Omega; \Gamma_0); \tag{2.7}$$

and for

$$\Gamma_0 = \Gamma_1 = \Gamma, \qquad H_0^2(\Omega; \Gamma, \Gamma) = H_0^2(\Omega). \tag{2.8}$$

$D'(\Omega)$ is the space of Schwartz distributions on $D(\Omega)$, $D(\Omega)$ being the space of test functions [4, 1].

Remark (2.1).The closed subspaces $H^2(\Omega;\ \Gamma_0,\ \Gamma_1)$, $H^2(\Omega) \cap H_0^1(\Omega; \Gamma_0)$ (resp. $H_0^1(\Omega; \Gamma_0)$) equipped with the subspace topology induced by $H^2(\Omega)$ (resp. $H^1(\Omega)$) are Hilbert spaces with the following inclusions:

$$H_0^2(\Omega) \subset H_0^2(\Omega; \Gamma_0, \Gamma_1) \subset H^2(\Omega) \cap H_0^1(\Omega; \Gamma_0) \subset H^2(\Omega); \tag{2.9}$$

$$H_0^1(\Omega) \subset H_0^1(\Omega; \Gamma_0) \subset H^1(\Omega). \tag{2.10}$$

3. Boundary Value Problems of Fourth Order Elliptic Equations

We consider the fourth order elliptic equations with variable coefficients [2]

$$\Lambda u = f \qquad \text{in} \quad \Omega, \tag{3.1}$$

where $f \in L^2(\Omega)$ is the given function, $u = u(x_1, x_2)$ is the unknown function,

$$(Au)(x) \equiv \frac{\partial^2}{\partial x_k \, \partial x_l}\left(a_{ijkl}\frac{\partial^2 u}{\partial x_i \, \partial x_j}\right)(x)$$
$$= (a_{ijkl} u_{,ij})_{,kl}(x) \tag{3.2}$$

(in (3.2) and also in the sequel, the Einstein summation convention has been followed), and the coefficients a_{ijkl} satisfy the following conditions: $\forall i, j, k, l = 1, 2,$

(A1) $a_{ijkl} \in C^0(\bar{\Omega})$,

(A2) $a_{ijkl}(x) = a_{klij}(x) = a_{lkij}(x) = a_{lkji}(x) \; \forall x \in \bar{\Omega}$,

(A3) $\exists \alpha_0 > 0$ such that $\forall \xi = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22})$ with $\xi_{12} = \xi_{21}$

$$a_{ijkl}(x)\, \xi_{ij} \xi_{kl} \geqslant \alpha_0 \, \|\xi\|^2_{\mathbb{R}^4} \qquad \forall x \in \bar{\Omega}.$$

Remark (3.1). Condition (A2) could have been replaced by a less restrictive condition $\forall i, j, k, l = 1, 2,$

$$a_{ijkl} = a_{klij}, \tag{3.3}$$

since, if for example $a_{ijkl} \neq a_{jikl}$ or $a_{ijkl} \neq a_{ijlk}$ for some $i, j, k, l = 1, 2$, we can always define new coefficients $\bar{a}_{ijkl}$ for which (A2) holds; i.e., $\forall i, j, k, l = 1, 2,$

$$\bar{a}_{ijkl} = (a_{ijkl} + a_{jikl} + a_{jilk} + a_{ijlk})/4 \tag{3.4}$$

such that

$$\bar{a}_{ijkl}(x) = \bar{a}_{klij}(x) = \bar{a}_{jilk}(x) = \bar{a}_{ijlk}(x) \quad \forall x \in \bar{\Omega}, \tag{3.5}$$

and

$$(Au)(x) = (a_{ijkl} u_{,ij})_{,kl}(x)$$
$$= (\bar{a}_{ijkl} u_{,ij})_{,kl}(x) \qquad \forall x \in \Omega. \tag{3.6}$$

Remark (3.2). For some sufficient conditions for (A3) to hold under an additional assumption that $a_{ijkl} \geqslant 0 \; \forall i, j, k, l = 1, 2$, see [2].

Boundary conditions. Let $\{\Gamma_i\}_{i=0}^3$ denote open subsets of Γ, on which boundary conditions involving derivatives of u with highest order $i = 0, 1, 2, 3$ are prescribed, such that

$$\Gamma_0 \neq \phi, \bar{\Gamma}_0 \cup \bar{\Gamma}_3 = \Gamma, \Gamma_0 \cap \Gamma_3 = \phi, \bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \Gamma, \Gamma_1 \cap \Gamma_2 = \phi, \tag{3.7}$$

where the condition $\Gamma_0 \neq \phi$ indicates that the boundary condition corresponding to $i=0$ is always prescribed on Γ_0. But in (3.7) it is possible to have $\Gamma_j=\phi$ for some $j=1, 2, 3$ indicating that boundary conditions involving derivatives of u having order "j" are absent.

In order to fix our ideas, we consider the following types of homogeneous boundary conditions on Γ_0, Γ_1, Γ_2, and Γ_3, some or all of which u must satisfy,

$$\text{(i) } u|_{\Gamma_0}=0; \quad \text{(ii) } (\partial u/\partial n)|_{\Gamma_1}=0; \quad \text{(iii) } M_n|_{\Gamma_2}=0; \quad \text{(iv) } K_n|_{\Gamma_3}=0, \tag{3.8}$$

where Γ_0, Γ_1, Γ_2, Γ_3 satisfy condition (3.7),

$$\begin{gathered} M_n=\psi_{ij}n_in_j; \quad K_n=\frac{\partial M_{nt}}{\partial t}+Q_n \text{ with } M_{nt}=\psi_{ij}n_it_j,\ Q_n=(\psi_{ij})_{,i}\,n_j; \\ \psi_{ij}=a_{ijkl}u_{,kl},\ 1\leqslant i, j\leqslant 2,\ \psi_{ij}=\psi_{ji}; \end{gathered} \tag{3.9}$$

$\mathbf{n}=(n_1, n_2)$ is the exterior unit vector normal to Γ, $\mathbf{t}=(t_1, t_2)=(n_2, -n_1)$ is the unit vector tangent to Γ; $\mathbf{n}$, $\mathbf{t}$ are defined a.e. on Γ.

Remark (3.3). In bending problems of elastic anisotropic/orthotropic/isotropic plates, the Neumann (natural) boundary conditions (iii) and (iv) in (3.8) denote the normal bending moment M_n at the boundary Γ_2 and the Kirchhoff transverse force K_n at the boundary Γ_3 respectively.

Green's formula. First of all we will derive Green's formula for sufficiently smooth $\Psi=(\psi_{ij})$ with $\psi_{ij}=a_{ijkl}u_{,kl}$. For this we define the spaces $D(\mathbb{R}^2)=\{\phi\colon \phi\in C^\infty(\mathbb{R}^2)$, supp ϕ is a compact subset of $\mathbb{R}^2\}$;

$$\begin{aligned} &D(\bar\Omega)=\{v\colon v=\phi|_{\bar\Omega} \text{ with } \phi\in D(\mathbb{R}^2)\}; \\ &\mathbb{D}(\bar\Omega)=\{\Phi\colon \Phi=(\phi_{ij})_{1\leqslant i,j\leqslant 2} \text{ with } \phi_{ij}=\phi_{ji}\in D(\bar\Omega)\}. \end{aligned} \tag{3.10}$$

PROPOSITION (3.1). *Let Γ be a sufficiently smooth boundary, i.e., $\boldsymbol{\phi}_{x_0}\colon s\in[0, L)\subset\mathbb{R}\mapsto\boldsymbol{\phi}_{x_0}(s)=x\in\Gamma\subset\mathbb{R}^2$ is a sufficiently smooth mapping with $\boldsymbol{\phi}_{x_0}(0)=x_0\in\Gamma$, "$s$" being the arc length of the oriented curve Γ measured from the point $x_0\in\Gamma$, whose total length is L.* (3.11)

Then, $\forall\Psi=(\psi_{ij})\in\mathbb{D}(\bar\Omega)$, $\forall v\in H^2(\Omega)$, the following Green's formula holds,

$$\begin{aligned} &\int_\Omega \psi_{ij,ij}v\,d\Omega-\int_\Omega \psi_{ij}v_{,ij}\,d\Omega \\ &\qquad=\int_\Gamma K_n(\Psi)v\,ds-\int_\Gamma M_n(\Psi)\frac{\partial v}{\partial n}\,ds, \end{aligned} \tag{3.12}$$

where $K_n(\Psi)$ and $M_n(\Psi)$ are defined by (3.9), $v|_\Gamma=\gamma_0 v\in H^{3/2}(\Gamma)\subset C^0(\Gamma)$, $\partial v/\partial n|_\Gamma=\gamma_1 v\in H^{1/2}(\Gamma)\subset L^2(\Gamma)$.

Proof. $\forall \Psi = (\psi_{ij}) \in \mathbb{D}(\bar{\Omega})$, $\forall v \in H^2(\Omega)$, the *usual* Green's formula has the form:

$$\int_\Omega \psi_{ij,ij} v \, d\Omega - \int_\Omega \psi_{ij} v_{,ij} \, d\Omega$$
$$= \int_\Gamma (\psi_{ij,i} n_j) v \, ds - \int_\Gamma (\psi_{ij} v_{,i} n_j) \, ds. \tag{3.13}$$

Since Γ is sufficiently smooth, $\partial v/\partial x_i = (\partial v/\partial n) n_i + (\partial v/\partial t) t_i$ $(i = 1, 2)$

$$\begin{aligned} \Rightarrow & \int_\Gamma \psi_{ij} v_{,i} n_j \, ds \\ &= \int_\Gamma \psi_{ij} \left(\frac{\partial v}{\partial n} n_i + \frac{\partial v}{\partial t} t_i \right) n_j \, ds \\ &= \int_\Gamma M_n(\Psi) \frac{\partial v}{\partial n} \, ds + \int_\Gamma M_{nt}(\Psi) \frac{\partial v}{\partial t} \, ds \\ &= \int_\Gamma M_n(\Psi) \frac{\partial v}{\partial n} \, ds + \int_\Gamma \frac{\partial}{\partial t} (M_{nt}(\Psi) v) \, ds - \int_\Gamma \frac{\partial}{\partial t} M_{nt}(\Psi) v \, ds \end{aligned} \tag{3.14}$$

$$\begin{aligned} \Rightarrow & \int_\Gamma (\psi_{ij,i} n_j) v \, ds - \int_\Gamma \psi_{ij} v_{,i} n_j \, ds \\ &= \int_\Gamma \left[\frac{\partial}{\partial t} M_{nt}(\Psi) + Q_n(\Psi) \right] v \, ds - \int_\Gamma M_n(\Psi) \frac{\partial v}{\partial n} \, ds, \end{aligned}$$

since for sufficiently smooth Γ,

$$\int_\Gamma \frac{\partial}{\partial t} [M_{nt}(\Psi) v] \, ds = [M_{nt}(\Psi) v](\boldsymbol{\phi}_{x_0}(s))|_{s=0}^{s=L} = 0. \tag{3.15}$$

Now, result (3.12) follows from (3.13), (3.14), and (3.9).

Remark (3.4). When Λ is the thin elastic plate bending operator and $v \in H^2(\Omega)$ is a variationally admissible displacement field, the line integrals in (3.12) denote (virtual) work done by the Kirchhoff force $K_n(\Psi)$ and the normal bending moment $M_n(\Psi)$ at the boundary Γ. Then, physical considerations of thin plate bending theory rule out the possibility of a simultaneous prescription of the *homogeneous* essential boundary condition $u|_{\Gamma_0} = 0$ (resp. $\partial u/\partial n|_{\Gamma_1} = 0$) and the *homogeneous* natural boundary condition $K_n(\Psi)|_{\Gamma_0} = 0$ (resp. $M_n(\Psi)|_{\Gamma_1} = 0$) on the same subset Γ_0 (resp. Γ_1) of Γ. Hence, both v (which must satisfy all essential boundary conditions) and $K_n(\Psi)$ (resp. both $\partial v/\partial n$ and $M_n(\Psi)$) with $\Psi \in \mathbb{D}(\bar{\Omega})$ *cannot* vanish simultaneously on any part of Γ, and this suggests that one write

$\int_\Gamma K_n(\Psi)v\,ds$ with $\Gamma=\bar{\Gamma}_0\cup\bar{\Gamma}_3$, $\Gamma_0\cap\Gamma_3=\phi$ (resp. $\int_\Gamma M_n(\Psi)(\partial v/\partial n)\,ds$ with $\Gamma=\bar{\Gamma}_1\cup\bar{\Gamma}_2$, $\Gamma_1\cap\Gamma_2=\phi$). Now, $v|_{\Gamma_0}=0$ with $\Gamma_0\neq\phi$, $K_n(\Psi)|_{\Gamma_3}=0$ with $\Gamma_0\cap\Gamma_3=\phi$, $\bar{\Gamma}_0\cup\bar{\Gamma}_3=\Gamma$ (resp. $\partial v/\partial n|_{\Gamma_1}=0$, $M_n(\Psi)|_{\Gamma_2}=0$ with $\Gamma_1\cap\Gamma_2=\phi$, $\bar{\Gamma}_1\cup\bar{\Gamma}_2=\Gamma$) will imply that $\int_\Gamma K_n(\Psi)v\,ds=0$ (resp. $\int_\Gamma M_n(\Psi)(\partial v/\partial n)\,ds=0$). These physical considerations have motivated the introduction of the homogeneous boundary conditions (3.8) in order to bring a large class of boundary value problems of practical interest and importance under a general treatment.

Remark (3.5). If $\boldsymbol{\phi}_{x0}$: $[0,L)\subset\mathbb{R}\to\Gamma$ defined in (3.11) is of C^1-class piecewise, i.e., $\exists$ a finite number of points $0<s_1<s_2<\cdots<s_n<L$ on $[0,L)$ such that $(d/ds)\boldsymbol{\phi}_{x0}$ has finite jump discontinuities at s_k, $1\leqslant k\leqslant n$, then $\forall\Psi\in\mathbb{D}(\bar{\Omega})$, $\forall v\in H^2(\Omega)$, (3.15) will *not* hold and we have

$$\int_\Gamma\frac{\partial}{\partial t}\left[M_{nt}(\Psi)v\right]ds$$

$$=-\sum_{i=1}^{n}\left[J(M_{nt})(\boldsymbol{\phi}_{x0}(s_i))\,v(\boldsymbol{\phi}_{x0}(s_i))\right],\tag{3.16}$$

where $J(M_{nt})(\boldsymbol{\phi}_{x0}(s_i))=$ jump of $M_{nt}(\Psi)$ at $\boldsymbol{\phi}_{x0}(s_i)=M_{nt}(\Psi)[\boldsymbol{\phi}_{x0}(s_i^+)]-M_{nt}(\Psi)[\boldsymbol{\phi}_{x0}(s_i^-)]$, $v(\boldsymbol{\phi}_{x0}(s_i))=v(\boldsymbol{\phi}_{x0}(s_i^\pm))$, $1\leqslant i\leqslant n$, since $H^2(\Omega)\subset C^0(\bar{\Omega})$. Then Green's formula has the following form: $\forall\Psi\in\mathbb{D}(\bar{\Omega})$, $\forall v\in H^2(\Omega)$,

$$\int_\Omega\psi_{ij,ij}v\,d\Omega-\int_\Omega\psi_{ij}v_{,ij}\,d\Omega=\int_\Gamma K_n(\Psi)v\,ds-\int_\Gamma M_n(\Psi)\frac{\partial v}{\partial n}\,ds$$

$$+\left\{\sum_{i=1}^{n}\left[J(M_{nt})(\boldsymbol{\phi}_{x0}(s_i))\,v(\boldsymbol{\phi}_{x0}(s_i))\right]\right\}.\tag{3.17}$$

But if $v(\boldsymbol{\phi}_{x0}(s_i))=0$, $1\leqslant i\leqslant n$ (see Example (7.6)), then the expression in the curly brackets in (3.17) vanishes and Green's formula (3.17) reduces to (3.12).

For $u\in H^2(\Omega)$, $M_n(\Psi)|_\Gamma$ and $K_n(\Psi)|_\Gamma$ with $\Psi=(\psi_{ij})$, $\psi_{ij}=a_{ijkl}u_{,kl}$, are *not* defined in general, and Green's formula (3.12) loses its meaning. Hence, we consider the space $H^2(\Lambda,\Omega)$ with Λ defined by (3.2),

$$H^2(\Lambda,\Omega)=\{v\colon v\in H^2(\Omega),\ \Lambda v\in L^2(\Omega)\}\tag{3.18}$$

equipped with the graph norm $\|\cdot\|_{2,\Lambda,\Omega}$ and innerproduct $\langle\cdot,\cdot\rangle_{2,\Lambda,\Omega}$

$$\|v\|^2_{2,\Lambda,\Omega}=\|v\|^2_{2,\Omega}+\|\Lambda v\|^2_{0,\Omega};$$

$$\langle u,v\rangle_{2,\Lambda,\Omega}=\langle u,v\rangle_{2,\Omega}+\langle\Lambda u,\Lambda v\rangle_{0,\Omega},$$

which is a Hilbert space with the continuous imbedding $H^2(\Lambda,\Omega)\subsetneq H^2(\Omega)$. Now, we will extend Green's formula (3.12) to functions $u \in H^2(\Lambda, \Omega)$ and $v \in H^2(\Omega)$, for which a complete proof is essential and given here. For the outline of the proof we refer the reader to [3], in which a detailed proof for the case of the Laplace operator Δ has been given. Introduce the space

$$\mathbb{L}^2(\Omega) = \{\Phi\colon \Phi = (\phi_{ij})_{1 \leqslant i,j \leqslant 2} \text{ with } \phi_{ij} = \phi_{ji} \in L^2(\Omega);\ \phi_{ij,ij} \in L^2(\Omega)\} \tag{3.19}$$

equipped with the norm $|||\cdot|||_{0,\Omega}$ and inner produce $[\![\cdot,\cdot]\!]_{0,\Omega}$:

$$\begin{aligned} |||\Phi|||^2_{0,\Omega} &= \|\Phi\|^2_{0,\Omega} + \|\phi_{ij,ij}\|^2_{0,\Omega} \\ &= \int_\Omega (\phi_{11}^2 + 2\phi_{12}^2 + \phi_{22}^2)\, d\Omega \\ &\quad + \int_\Omega (\phi_{11,11} + 2\phi_{12,12} + \phi_{22,22})^2\, d\Omega, \\ [\![\Psi, \Phi]\!]_{0,\Omega} &= \langle\!\langle \Psi, \Phi \rangle\!\rangle_{0,\Omega} + \langle \psi_{ij,ij}, \phi_{ij,ij} \rangle_{0,\Omega} \qquad \text{with} \\ \langle\!\langle \Psi, \Phi \rangle\!\rangle_{0,\Omega} &= \langle\!\langle (\psi_{ij}), (\phi_{ij}) \rangle\!\rangle_{0,\Omega} \\ &= \int_\Omega (\psi_{11}\phi_{11} + 2\psi_{12}\phi_{12} + \psi_{22}\phi_{22})\, d\Omega, \end{aligned}$$

which is also a Hilbert space.

For fixed $\Psi = (\psi_{ij}) \in \mathbb{L}^2(\Omega)$, define $T_\Psi\colon H^{3/2}(\Gamma) \times H^{1/2}(\Gamma) \to \mathbb{R}$ by

$$\begin{gathered} T_\Psi(\chi) = \int_\Omega \psi_{ij,ij}\omega(\chi)\, d\Omega - \int_\Omega \psi_{ij}(\omega(\chi))_{,ij}\, d\Omega \\ \forall \chi = (\chi_0, \chi_1) \text{ with } \chi_0 \in H^{3/2}(\Gamma),\ \chi_1 \in H^{1/2}(\Gamma), \end{gathered} \tag{3.20}$$

where ω: $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^2(\Omega)$ is a linear, continuous operator defined by

$$\omega\colon \chi = (\chi_0, \chi_1) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma) \mapsto \omega(\chi) \in H^2(\Omega) \qquad \text{with}$$

$$\gamma_0\omega(\chi) = \omega(\chi)|_\Gamma = \chi_0,\ \gamma_1\omega(\chi) = \frac{\partial}{\partial n}\omega(\chi)|_\Gamma = \chi_1, \tag{3.21}$$

$$\begin{aligned} \|\omega(\chi)\|_{2,\Omega} &\leqslant C\, \|\chi\|_{H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)} \\ &= C(\|\chi_0\|^2_{H^{3/2}(\Gamma)} + \|\chi_1\|^2_{H^{1/2}(\Gamma)})^{1/2}, \end{aligned} \tag{3.22}$$

$\omega(\chi) \in H^2(\Omega)$ satisfying (3.21), (3.22) is *not* unique.

LEMMA (3.1). (i) *$T_\Psi(\chi)$ in* (3.20) *does* not *depend on the choice of $\omega(\chi) \in H^2(\Omega)$ satisfying* (3.21), (3.22).

(ii) *T_Ψ is a linear, continuous functional on $H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$ with*

$$\|T_\Psi\|_{H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma)} \leqslant C\,|||\Psi|||_{\mathbb{L}^2(\Omega)} \qquad (C>0) \tag{3.23}$$

and, consequently, can be written as follows: $\forall\chi=(\chi_0,\chi_1)\in H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$,

$$\begin{aligned} T_\Psi(\chi) &= [h,\chi_0]_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)} \\ &\quad - [g,\chi_1]_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \qquad \text{with} \quad g\in H^{-1/2}(\Gamma),\ h\in H^{-3/2}(\Gamma). \end{aligned} \tag{3.24}$$

$$\text{(iii) } \Psi=(\psi_{ij})\in\mathbb{D}(\bar\Omega)\Rightarrow\begin{cases} K_n(\Psi)|_\Gamma = h\in H^{-3/2}(\Gamma), & (3.25)\\ M_n(\Psi)|_\Gamma = g\in H^{-1/2}(\Gamma). & (3.26)\end{cases}$$

Proof. (i) Suppose $\exists\omega^1,\ \omega^2\in H^2(\Omega)$ such that $\gamma_0\omega^i=\chi_0$, $\gamma_1\omega^i=\chi_1$ $(i=1,2)$. Set $W^1=(\omega^1_{,ij})$, $W^2=(\omega^2_{,ij})$. We are to show that $T_\Psi(\chi)=\langle\psi_{ij,ij},\omega^i\rangle_{0,\Omega}-\langle\langle\Psi,W^i\rangle\rangle_{0,\Omega}$ $(i=1,2)$ or equivalently that $\langle\psi_{ij,ij},\omega^1-\omega^2\rangle_{0,\Omega}=\langle\langle\Psi,W^1-W^2\rangle\rangle_{0,\Omega}$. But

$$\begin{aligned}\gamma_0(\omega^1-\omega^2)=\gamma_1(\omega^1-\omega^2)=0 &\Rightarrow(\omega^1-\omega^2)\in H^2_0(\Omega)\\ &\Rightarrow\langle\psi_{ij,ij},\omega^1-\omega^2\rangle_{0,\Omega}=\langle\langle\Psi,W^1-W^2\rangle\rangle_{0,\Omega}.\end{aligned}$$

(ii) The linearity of T_Ψ is obvious from (3.20), and its continuity follows from: $\forall\chi=(\chi_0,\chi_1)\in H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$, satisfying (3.21), (3.22),

$$\begin{aligned}|T_\Psi(\chi)| &\leqslant \|\psi_{ij,ij}\|\ \|\omega(\chi)\|_{0,\Omega}+\|\Psi\|_{0,\Omega}\ \|((\omega(\chi))_{,ij})\|_{0,\Omega}\\ &\leqslant (\|\Psi\|^2_{0,\Omega}+\|\psi_{ij,ij}\|^2_{0,\Omega})^{1/2}\\ &\qquad\times(\|\omega(\chi)\|^2_{0,\Omega}+\|((\omega(\chi))_{,ij})\|^2_{0,\Omega})^{1/2}\\ &\leqslant |||\Psi|||_{0,\Omega}\ \|\omega(\chi)\|_{2,\Omega}\\ &\leqslant (C\,|||\Psi|||_{0,\Omega})\ \|\chi\|_{H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)} \quad \text{(by (3.22))}\\ &\Rightarrow T_\Psi\in H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma) \qquad \text{with}\\ &\qquad \|T_\Psi\|_{H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma)}\leqslant C\,|||\Psi|||_{0,\Omega} \qquad \forall\text{ fixed }\Psi\in\mathbb{L}^2(\Omega).\end{aligned}$$

(iii) Let $\Psi=(\psi_{ij})\in\mathbb{D}(\bar\Omega)\subset\mathbb{L}^2(\Omega)$. Then $\forall\chi=(\chi_0,\chi_1)\in H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$ satisfying (3.21), (3.22), we have from (3.20) and (3.12)

$$\begin{aligned}T_\Psi(\chi)&=\int_\Omega\psi_{ij,ij}\,\omega(\chi)\,d\Omega-\int_\Omega\psi_{ij}(\omega(\chi))_{,ij}\,d\Omega\\ &=\int_\Gamma K_n(\Psi)\,\gamma_0(\omega(\chi))\,ds-\int_\Gamma M_n(\Psi)\,\gamma_1(\omega(\chi))\,ds\\ &=\int_\Gamma K_n(\Psi)\,\chi_0\,ds-\int_\Gamma M_n(\Psi)\,\chi_1\,ds \quad \text{(by (3.21))}.\end{aligned}\tag{3.27}$$

Then, the results (3.25), (3.26) follow from a comparison of (3.27) and (3.24). Now, (3.25), (3.26) allow us to define an operator $\boldsymbol{\delta}$: $\mathbb{D}(\bar{\Omega}) \to H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$ by

$$\forall \Psi \in \mathbb{D}(\bar{\Omega}), \qquad \boldsymbol{\delta}\Psi = T_{\Psi} \in H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \tag{3.28}$$

such that

$$\begin{aligned}[\boldsymbol{\delta}\Psi, \boldsymbol{\chi}] = [T_{\Psi}, \boldsymbol{\chi}] &= [K_n(\Psi), \chi_0]_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} \\ &\quad - [M_n(\Psi), \chi_1]_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} \\ \forall \boldsymbol{\chi} &= (\chi_0, \chi_1) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma).\end{aligned} \tag{3.29}$$

PROPOSITION (3.2). (i) *$\boldsymbol{\delta}$ defined by (3.28), (3.29) is linear and continuous on $\mathbb{D}(\bar{\Omega})$ in the norm of $\mathbb{L}^2(\Omega)$ with*

$$\begin{aligned}\|\boldsymbol{\delta}\Psi\|_{H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)} &= \|T_{\Psi}\|_{H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)} \\ &\leqslant C \, |||\Psi|||_{0,\Omega} \qquad \forall \Psi \in \mathbb{D}(\bar{\Omega}).\end{aligned} \tag{3.30}$$

(ii) *Green's formula* (3.12) *can be rewritten as follows*: $\forall \Psi \in \mathbb{D}(\bar{\Omega})$, $\forall v \in H^2(\Omega)$,

$$\begin{aligned}&\langle \psi_{ij,ij}, v \rangle_{0,\Omega} - \langle\!\langle (\psi_{ij}), (v_{,ij}) \rangle\!\rangle_{0,\Omega} \\ &\quad = [\boldsymbol{\delta}\Psi, (\gamma_0 v, \gamma_1 v)] \\ &\quad = [K_n(\Psi), \gamma_0 v]_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} \\ &\qquad - [M_n(\Psi), \gamma_1 v]_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}.\end{aligned} \tag{3.31}$$

THEOREM (3.1). *Let Γ be sufficiently smooth* (3.11).

(i) $\mathbb{D}(\bar{\Omega})$ *is dense in* $\mathbb{L}^2(\Omega)$.

(ii) *The linear, continuous operator* $\boldsymbol{\delta}$: $\mathbb{D}(\bar{\Omega}) \to H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$ *defined by* (3.28), (3.29) *can be given a unique continuous extension to a linear, continuous operator, still denoted by the same symbol $\boldsymbol{\delta}$, from $\mathbb{L}^2(\Omega)$ to $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$ such that* $\forall \Psi \in \mathbb{L}^2(\Omega)$, $\forall \boldsymbol{\chi} = (\chi_0, \chi_1) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$,

$$[\boldsymbol{\delta}\Psi, \boldsymbol{\chi}] = [K_n(\Psi), \chi_0]_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} - [M_n(\Psi), \chi_1]_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} \tag{3.32}$$

and the following Green's formula holds: $\forall \Psi \in \mathbb{L}^2(\Omega)$, $\forall v \in H^2(\Omega)$

$$\begin{aligned}&\langle \psi_{ij,ij}, v \rangle_{0,\Omega} - \langle\!\langle (\psi_{ij}), (v_{,ij}) \rangle\!\rangle_{0,\Omega} \\ &\quad = [K_n(\Psi), \chi_0]_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} - [M_n(\Psi), \chi_1]_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}.\end{aligned} \tag{3.33}$$

Proof. Assume for the moment that (i) holds. Then (ii) follows from (i), i.e., the density of $\mathbb{D}(\bar{\Omega})$ in $\mathbb{L}^2(\Omega)$, which will be proved now, gives (3.32) and (3.33).

(i) Let l be a continuous linear functional on $\mathbb{L}^2(\Omega)$. Then, by the Riesz representation theorem, $\exists$ a unique $\Psi=(\psi_{ij})\in\mathbb{L}^2(\Omega)$ such that $\forall\Phi\in\mathbb{L}^2(\Omega)$,

$$\begin{aligned} l(\Phi) &= [\![\Psi, \Phi]\!]_{0,\Omega} \\ &= \langle\!\langle \Psi, \Phi\rangle\!\rangle_{0,\Omega} + \langle q, \phi_{ij,ij}\rangle_{0,\Omega} \qquad \text{with} \quad q=\psi_{ij,ij}. \end{aligned} \tag{3.34}$$

From the Hahn–Banach theorem, the density result will follow, if we can prove that $l(\Phi)=0\ \forall\Phi\in\mathbb{D}(\bar{\Omega})\Rightarrow l=0$. Assume that

$$\begin{aligned} l(\Phi) &= \langle\!\langle (\psi_{ij}), (\phi_{ij})\rangle\!\rangle_{0,\Omega} + \langle q, \phi_{ij,ij}\rangle_{0,\Omega} \\ &= 0 \qquad \forall\Phi\in\mathbb{D}(\bar{\Omega}). \end{aligned} \tag{3.35}$$

Then, $l(\Phi)=0\ \forall\Phi\in\mathbb{D}(\Omega)=\{\Phi\colon \Phi=(\phi_{ij})_{1\leqslant i,j\leqslant 2}$ with $\phi_{ij}=\phi_{ji}\in D(\Omega)\}\subset\mathbb{D}(\bar{\Omega}) \Rightarrow \sum_{i,j=1}^2\{[\psi_{ij}, \phi_{ij}]_{D'(\Omega)\times D(\Omega)}+[q_{,ij}, \phi_{ij}]_{D'(\Omega)\times D(\Omega)}\} = 0\ \forall\Phi=(\phi_{ij})\in\mathbb{D}(\Omega) \Rightarrow \psi_{ij}+q_{,ij}=0$ in $D'(\Omega)$, $1\leqslant i, j\leqslant 2\Rightarrow q_{,ij}=-\psi_{ij}\in L^2(\Omega)\ \forall i$, $j=1, 2\Rightarrow q\in H^2(\Omega)$. Now we will show that $q\in H_0^2(\Omega)$. For this we will first show that (a) $q|_\Gamma=0$ and then (b) $\partial q/\partial n|_\Gamma=0$.

(a) $\forall v\in D(\bar{\Omega})$ with $v_{,ij}|_\Gamma=0$ $(i,\ j=1,\ 2)$, $q\in H^2(\Omega)$, the usual Green's formula gives:

$$\begin{aligned} \langle\!\langle (\psi_{ij}), (v_{,ij})\rangle\!\rangle_{0,\Omega} &= -\langle\!\langle (q_{,ij}), (v_{,ij})\rangle\!\rangle_{0,\Omega} \\ &= \int_\Omega \operatorname{grad} q\cdot[\operatorname{grad}(v_{,11})+\operatorname{grad}(v_{,22})]\, d\Omega \\ &= -\int_\Omega q(v_{,ij})_{,ij}\, d\Omega - \int_\Gamma q\frac{\partial}{\partial n}(\Delta v)\, ds. \end{aligned}$$

From (3.35), $l((v_{,ij}))=-\langle\!\langle (q_{,ij}), (v_{,ij})\rangle\!\rangle_{0,\Omega}+\langle q, (v_{,ij})_{,ij}\rangle_{0,\Omega}=0\ \forall v\in D(\bar{\Omega})$ with $v_{,ij}|_\Gamma=0 \Rightarrow \int_\Gamma q(\partial/\partial n)(\Delta v)\, ds=0\ \forall v\in D(\bar{\Omega})$ with $v_{,ij}|_\Gamma=0\Rightarrow q|_\Gamma=0$.

(b) Choose $\Phi^*=(\phi^*_{ij})\in\mathbb{D}(\bar{\Omega})$ with $\phi^*_{ij}=(\phi_{i,j}+\phi_{j,i})/2$, $\phi_i\in D(\bar{\Omega})$ $(i=1, 2)$. Then

$$\begin{aligned} \langle\!\langle (\psi_{ij}), (\phi^*_{ij})\rangle\!\rangle_{0,\Omega} &= -\langle\!\langle (q_{,ij}), (\phi^*_{ij})\rangle\!\rangle_{0,\Omega} \\ &= -\int_\Omega \operatorname{div}(q_{,1}\operatorname{grad}\phi_1)\, d\Omega \\ &\quad -\int_\Omega \operatorname{div}(q_{,2}\operatorname{grad}\phi_2)\, d\Omega \\ &\quad +\int_\Omega [q_{,1}(\Delta\phi_1)+q_{,2}(\Delta\phi_2)]\, d\Omega \end{aligned}$$

$$= -\int_\Gamma \left(q_{,1}\frac{\partial\phi_1}{\partial n} + q_{,2}\frac{\partial\phi_2}{\partial n}\right) ds$$

$$+\int_\Gamma q(\Delta\phi_1, \Delta\phi_2)\cdot\mathbf{n}\, ds$$

$$-\int_\Omega q\left[\frac{\partial}{\partial x_1}(\Delta\phi_1) + \frac{\partial}{\partial x_2}(\Delta\phi_2)\right] d\Omega$$

$$= -\int_\Omega q\phi^*_{ij,ij}\, d\Omega$$

$$-\int_\Gamma \left(q_{,1}\frac{\partial\phi_1}{\partial n} + q_{,2}\frac{\partial\phi_2}{\partial n}\right) ds, \quad \text{since } q|_\Gamma = 0,$$

$$\phi^*_{ij,ij} = \frac{\partial}{\partial x_1}(\Delta\phi_1) + \frac{\partial}{\partial x_2}(\Delta\phi_2).$$

Then, from (3.35) with $\psi_{ij} = -q_{,ij}\ \forall i, j = 1, 2$, $l(\Phi^*) = 0$

$$\Rightarrow \int_\Gamma \left(q_{,1}\frac{\partial\phi_1}{\partial n} + q_{,2}\frac{\partial\phi_2}{\partial n}\right) ds = 0 \qquad \forall\phi_1, \phi_2 \in D(\bar\Omega)$$

$$\Rightarrow q_{,1}|_\Gamma = 0 = q_{,2}|_\Gamma \Rightarrow \left.\frac{\partial q}{\partial n}\right|_\Gamma = 0, \qquad \text{i.e., } q \in H^2_0(\Omega).$$

Let $\{q_n\}$ be a sequence in $D(\Omega)$ such that $q_n \to q \in H^2_0(\Omega)$ as $n \to \infty$. Then,

$$\forall\Phi \in \mathbb{L}^2(\Omega),\ \lim_{n\to\infty} [-\langle\!\langle (q_{n,ij}), (\phi_{ij})\rangle\!\rangle_{0,\Omega} + \langle q_n, \phi_{ij,ij}\rangle_{0,\Omega}]$$

$$= -\langle\!\langle (q_{,ij}), (\phi_{ij})\rangle\!\rangle_{0,\Omega} + \langle q, \phi_{ij,ij}\rangle_{0,\Omega} = l(\Phi)$$

$$\Rightarrow \forall\Phi \in \mathbb{L}^2(\Omega),$$

$$l(\Phi) = \lim_{n\to\infty}\left[-\int_\Omega q_{n,ij}\phi_{ij}\, d\Omega + \int_\Omega q_{n,ij}\phi_{ij}\, d\Omega\right] = 0,$$

$$\Rightarrow l = 0,$$

from which the density result follows, and the theorem is proved. But

$$u \in H^2(\Lambda, \Omega) \Rightarrow \psi_{ij} = a_{ijkl}u_{,kl} \in L^2(\Omega),$$

$$\psi_{ij,ij} = (a_{ijkl}u_{,kl})_{,ij} = \Lambda u \in L^2(\Omega)$$

$$\Rightarrow \Psi = (\psi_{ij}) \in \mathbb{L}^2(\Omega) \qquad \text{with}$$

$$\psi_{ij} = a_{ijkl}u_{,kl} \qquad \forall i, j = 1, 2.$$

Now, we define a Neumann operator $\mathbf{v}$: $H^2(\Lambda, \Omega) \to H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$ as follows,

$$\begin{aligned} &\forall u \in H^2(\Lambda, \Omega) \qquad \text{with} \quad \psi_{ij} = a_{ijkl} u_{,kl}, \\ &\mathbf{v}u = \boldsymbol{\delta}\Psi \in H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \qquad (\Psi = (\psi_{ij}) \in \mathbb{L}^2(\Omega)) \end{aligned} \tag{3.36}$$

such that

$$\begin{aligned} [\mathbf{v}u, \chi] = [\boldsymbol{\delta}\Psi, \chi] &= [K_n(\Psi), \chi_0]_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} \\ &\quad - [M_n(\Psi), \chi_1]_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}, \end{aligned} \tag{3.37}$$

where $\boldsymbol{\delta}$: $\mathbb{L}^2(\Omega) \to H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is defined by (3.32),

$$K_n(\Psi)|_\Gamma = \left[\frac{\partial}{\partial t} (a_{ijkl} u_{,kl} n_i t_j) + (a_{ijkl} u_{,kl})_{,i} n_j \right] \Bigg|_\Gamma \in H^{-3/2}(\Gamma), \tag{3.38}$$

$$M_n(\Psi)|_\Gamma = (a_{ijkl} u_{,kl} n_i n_j)|_\Gamma \in H^{-1/2}(\Gamma). \tag{3.39}$$

Now, we state the final result on Green's formula.

THEOREM (3.2). *Let Γ be a sufficiently smooth boundary* (3.11). *The operator* $\mathbf{v}$: $H^2(\Lambda, \Omega) \to H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$ *defined in* (3.36)–(3.39) *is linear, continuous on $H^2(\Lambda, \Omega)$, and consequently, Green's formula has the following form*: $\forall u \in H^2(\Lambda, \Omega)$, $\forall v \in H^2(\Omega)$,

$$\begin{aligned} &\int_\Omega (\Lambda u) v \, d\Omega - \int_\Omega \psi_{ij} v_{,ij} \, d\Omega \\ &\quad = [K_n(\Psi), \gamma_0 v]_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} - [M_n(\Psi), \gamma_1 v]_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}, \end{aligned} \tag{3.40}$$

where $\Psi = (\psi_{ij})$ with $\psi_{ij} = a_{ijkl} u_{,kl}$ ($\forall i, j = 1, 2$), $K_n(\Psi)|_\Gamma$ and $M_n(\Psi)|_\Gamma$ are defined by (3.38) *and* (3.39) *respectively*,

$$\gamma_0 v = v|_\Gamma \in H^{3/2}(\Gamma), \qquad \gamma_1 v = \frac{\partial v}{\partial n}\bigg|_\Gamma \in H^{1/2}(\Gamma).$$

Proof. The linearity of $\mathbf{v}$ is obvious from (3.37)–(3.39). For its continuity, we have from (3.37): $\forall \chi \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$, $\forall u \in H^2(\Lambda, \Omega)$,

$$\begin{aligned} |[\mathbf{v}u, \chi]| = |[\boldsymbol{\delta}\Psi, \chi]| = |T_\Psi(\chi)| &\leqslant (C \, |||\Psi|||_{0,\Omega}) \\ &\quad \cdot \|\chi\|_{H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)} \quad \text{(by Lemma (3.1))} \end{aligned}$$

with $\Psi = (\psi_{ij}) \in \mathbb{L}^2(\Omega)$, $\psi_{ij} = a_{ijkl} u_{,kl} \Rightarrow \mathbf{v}u$ is continuous on $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ with

$$\|\mathbf{v}u\|_{H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)} \leqslant C \, |||\Psi|||_{0,\Omega}.$$

But

$$\psi_{ij}^2 = (a_{ij11}u_{,11} + 2a_{ij12}u_{,12} + a_{ij22}u_{,22})^2$$
$$\leqslant C_0[(u_{,11})^2 + (u_{,12})^2 + (u_{,22})^2]$$
$$\text{with some} \quad C_0 > 0 \text{ (by virtue of (A1))}$$
$$\Rightarrow |||\Psi|||^2_{\mathbb{L}^2(\Omega)} = \sum_{i,j=1}^{2} \int_\Omega \psi_{ij}^2 \, d\Omega + \|\Lambda u\|^2_{0,\Omega} \leqslant C_1 \|u\|^2_{2,\Omega}$$
$$+ \|\Lambda u\|^2_{0,\Omega} \leqslant C_2 \|u\|^2_{2,\Lambda,\Omega}.$$

Hence, $\|\mathbf{v}u\|_{H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma)} \leqslant C_3 \|u\|_{2,\Lambda,\Omega} \; \forall u \in H^2(\Lambda, \Omega) \Rightarrow \mathbf{v}$ is continuous on $H^2(\Lambda, \Omega)$.

Finally, Green's formula (3.40) follows from (3.33) and (3.37), since $\psi_{ij,ij} = \Lambda u$.

Now, we can define the boundary value problem (P) as follows: For given $f \in L^2(\Omega)$, find u such that

$$\text{(P)}: \quad \begin{aligned} &\Lambda u = f \text{ in } \Omega, \qquad \text{and} \\ &\text{(i) } u|_{\Gamma_0} = 0, \quad \text{(ii) } \left.\frac{\partial u}{\partial n}\right|_{\Gamma_1} = 0; \quad \text{(iii) } M_n|_{\Gamma_2} = 0; \quad \text{(iv) } K_n|_{\Gamma_3} = 0, \end{aligned} \tag{3.41}$$

where Γ_0, Γ_1, Γ_2, and Γ_3 satisfy conditions (3.7), and for some $j = 1, 2, 3$, Γ_j may be an empty set ϕ implying the absence of the corresponding boundary condition (see Examples (7.1), (7.2), (7.4)–(7.6)), Λ is defined in (3.2); M_n and K_n are defined in (3.9).

4. The Galerkin Variational Problem ($\mathrm{P_G}$)

Let V be a closed subspace of functions of $H^2(\Omega)$ which satisfy only the essential (Dirichlet) boundary conditions of type (i) and also of type (ii), whenever meas $(\Gamma_1) \neq 0$, in (3.41) such that

$$H_0^2(\Omega) \subset V \subset H^2(\Omega). \tag{4.1}$$

Then, V equipped with the subspace topology induced by $H^2(\Omega)$, i.e.,

$$\|v\|_V = \|v\|_{2,\Omega}, \quad \langle v, w \rangle_V = \langle v, w \rangle_{2,\Omega} \quad \forall v, w \in V, \tag{4.2}$$

is a Hilbert space.

Furthermore, we make the following important assumption on the space V:

(A4) $V \cap P_1(\bar{\Omega}) = \{0\}$,

where $P_1(\bar{\Omega})$ is the linear space of restrictions to $\bar{\Omega}$ of all polynomials of degree $\leqslant 1$ in the variables x_1 and x_2.

Remark (4.1). This assumption (A4) is *not* restrictive and will hold for a large class of boundary value problems of practical importance (see Examples (7.1)–(7.7)).

To the boundary value problem (P) in (3.41), we associate the Galerkin variational problem (P_G) defined by: For given $f \in L^2(\Omega)$, find $u \in V$ such that

$$(P_G): \quad a(u, v) = l(v) \qquad \forall v \in V, \tag{4.3}$$

where the symmetric, bilinear form $a(\cdot, \cdot)$: $V \times V \to \mathbb{R}$ and the linear form $l(\cdot)$: $V \to \mathbb{R}$ are given by

$$a(u, v) = \int_\Omega a_{ijkl} u_{,ij} v_{,kl} \, d\Omega \qquad \forall u, v \in V, \tag{4.4}$$

$$l(v) = \int_\Omega fv \, d\Omega \qquad \forall v \in V; \tag{4.5}$$

$a_{ijkl} = a_{ijkl}(x)$ satisfy conditions (A1)–(A3).

Remark (4.2). Formula (4.3) follows from (3.1), (3.2) and (3.40), (3.41).

5. Norm Equivalence in V

Theorem (5.1). *Suppose that assumption* (A4) *holds. Then, the mapping*

$$v \in V \to |v|_{2,\Omega} \tag{5.1}$$

defines a norm equivalent to the original norm (4.2) *in* V; *i.e.*, $\exists \beta_1, \beta_2 > 0$ *such that*

$$\beta_1 \|v\|_{2,\Omega} \leqslant |v|_{2,\Omega} \leqslant \beta_2 \|v\|_{2,\Omega} \qquad \forall v \in V. \tag{5.2}$$

Proof. The right-hand side inequality in (5.2) follows from the definition of the norm in $H^2(\Omega)$ with $\beta_2 = 1$. Now, we shall establish the left-hand side inequality in (5.2); i.e.,

$$\exists \beta_1 > 0 \text{ such that } \beta_1 \|v\|_{2,\Omega} \leqslant |v|_{2,\Omega} \quad \forall v \in V. \tag{5.3}$$

Suppose that (5.3) does not hold; i.e., $\forall \varepsilon > 0$ $\exists \tilde{v} \in V$ such that $|\tilde{v}|_{2,\Omega} < \varepsilon \, \|\tilde{v}\|_{2,\Omega}$

$$\Rightarrow \frac{|\tilde{v}|_{2,\Omega}}{\|\tilde{v}\|_{2,\Omega}} < \varepsilon \Rightarrow |v|_{2,\Omega} < \varepsilon \quad \text{with} \quad v = \tilde{v}/\|\tilde{v}\|_{2,\Omega},\ \|v\|_{2,\Omega} = 1.$$

Thus, $\exists$ a sequence $\{v_m\}_{m=1}^{\infty}$ in V such that

$$\|v_m\|_{2,\Omega} = 1 \qquad \text{and} \qquad |v_m|_{2,\Omega} < 1/m \qquad \forall m \in \mathbb{N}. \tag{5.4}$$

Hence, $\{v_m\}_{m=1}^{\infty}$ is a bounded sequence in $H^2(\Omega)$.

Again, since $H^2(\Omega) \subsetneq H^1(\Omega)$, the injection $\subsetneq$ being a compact operator by Kondraschev's theorem [1, 3], from this sequence $\{v_m\}_{m=1}^{\infty}$ in $H^1(\Omega)$, we can extract a convergent subsequence $\{v_{m_k}\}_{k=1}^{\infty}$ in $H^1(\Omega)$; i.e., $\exists v \in H^1(\Omega)$ such that

$$\begin{aligned} &\lim_{k \to \infty} \|v - v_{m_k}\|_{1,\Omega} = 0 \Rightarrow v_{m_k} \to v \quad \text{in} \quad L^2(\Omega) \quad \text{and} \\ &\frac{\partial}{\partial x_i} v_{m_k} \to \frac{\partial v}{\partial x_i} \quad \text{in} \quad L^2(\Omega) \text{ as } k \to \infty,\ i = 1, 2. \end{aligned} \tag{5.5}$$

From (5.4),

$$|v_{m_k}|_{2,\Omega} \to 0 \qquad \text{as} \quad k \to \infty \Rightarrow D^\alpha v_{m_k} \to 0 \quad \text{in } L^2(\Omega) \text{ as } k \to \infty \tag{5.6}$$

$\forall \alpha = (\alpha_1, \alpha_2)$ with $|\alpha| = \alpha_1 + \alpha_2 = 2$.

But D^α is continuous on $D'(\Omega)$ and $v_{m_k} \to v$ in $L^2(\Omega) \subset D'(\Omega)$ as $k \to \infty$

$$\begin{aligned} &\Rightarrow D^\alpha v_{m_k} \to D^\alpha v \quad \text{in} \quad D'(\Omega) \text{ as } k \to \infty,\ \forall \alpha = (\alpha_1, \alpha_2) \text{ with } |\alpha| = 2 \\ &\Rightarrow D^\alpha v = 0 \quad \text{in} \quad D'(\Omega)\ \forall \alpha = (\alpha_1, \alpha_2) \text{ with } |\alpha| = 2 \text{ (by (5.6))} \\ &\Rightarrow D^\alpha v = 0 \in L^2(\Omega) \qquad \forall \alpha = (\alpha_1, \alpha_2) \text{ with } |\alpha| = 2 \\ &\Rightarrow v \in H^2(\Omega) \qquad \text{with} \qquad D^\alpha v = 0\ \forall \alpha = (\alpha_1, \alpha_2) \text{ with } |\alpha| = 2. \end{aligned}$$

So, $\|v - v_{m_k}\|_{2,\Omega}^2 = |v - v_{m_k}|_{2,\Omega}^2 + \|v - v_{m_k}\|_{1,\Omega}^2 = |v_{m_k}|_{2,\Omega}^2 + \|v - v_{m_k}\|_{1,\Omega}^2 \to 0$ as $k \to \infty$, by (5.4) and (5.5),

$$\Rightarrow v_{m_k} \to v \quad \text{in} \quad H^2(\Omega) \text{ as } k \to \infty$$

$$\Rightarrow v \in V \quad \text{since } V \text{ is a closed subspace of } H^2(\Omega), \tag{5.7}$$

$$\text{and } \|v\|_{2,\Omega} = \lim_{k \to \infty} \|v_{m_k}\|_{2,\Omega} = 1 \qquad \text{(from (5.4))}. \tag{5.8}$$

Since

$$D^\alpha v = 0 \qquad \text{in} \quad \Omega\ \forall \alpha = (\alpha_1, \alpha_2) \text{ with } |\alpha| = 2,\ v \in P_1(\bar{\Omega}) \text{ [4]}. \tag{5.9}$$

Thus, from (5.7), (5.9), $v \in V \cap P_1(\bar{\Omega}) \Rightarrow v = 0$ by (A4) $\Rightarrow \|v\|_{2,\Omega} = 0$ which contradicts (5.8). Hence, our supposition that (5.3) does not hold is wrong and the result follows.

The norm equivalence in (5.1), (5.2) is based on the assumption that (A4) holds. Now, we will show that in many situations the assumption (A4) will hold.

PROPOSITION (5.1). *Let $\{a_i\}_{i=1}^3$ be any three noncollinear points on Γ and let V_0 be defined by*

$$V_0 = \{v: v \in H^2(\Omega), v(a_i) = 0, 1 \leqslant i \leqslant 3\}. \tag{5.10}$$

Then,

$$\forall V \text{ defined in } (4.1), (4.2) \text{ with } V \subset V_0, \tag{5.11}$$

assumption (A4) *holds.*

Remark (5.1). The definition of V_0 in (5.10) is meaningful by virtue of the imbedding $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$ [1, 3].

Proof. Let $v \in V \cap P_1(\bar{\Omega})$. Then, $v(x_1, x_2) = b_0 + b_1 x_1 + b_2 x_2$ with b_0, b_1, $b_2 \in \mathbb{R}$, $(x_1, x_2) \in \bar{\Omega}$. But $V \subset V_0 \Rightarrow v(a_i) = 0$ with noncollinear $a_i \in \Gamma$, $i = 1, 2, 3 \Leftrightarrow v = 0$ in $V \cap P_1(\bar{\Omega}) \Rightarrow V \cap P_1(\bar{\Omega}) = \{0\}$.

COROLLARY (5.1). *Suppose that $V \equiv H^2(\Omega) \cap H_0^1(\Omega)$ satisfying* (4.1), (4.2). *Then, assumption* (A4) *holds.*

PROPOSITION (5.2). *Let $V \equiv H_0^2(\Omega;\ \Gamma_0,\ \Gamma_1)$ defined in* (2.4) *with* meas $(\Gamma_0 \cap \Gamma_1) \neq 0$. *Then assumption* (A4) *holds.*

Proof. Let $v \in V \cap P_1(\bar{\Omega})$. Then, $v(x_1, x_2) = b_0 + b_1 x_1 + b_2 x_2$ with b_0, b_1, $b_2 \in \mathbb{R}$. From (2.4), we have $v|_{\Gamma_0} = 0$, $(\partial v/\partial n)|_{\Gamma_1} = 0$. But $v|_{\Gamma_0} = 0 \Rightarrow$ $(\partial v/\partial t)|_{\Gamma_0} = 0$. Hence, $(\partial v/\partial t) = (\partial v/\partial n) = 0$ on $\Gamma_0 \cap \Gamma_1$, meas$(\Gamma_0 \cap \Gamma_1) \neq 0$

$$\begin{aligned}
&\Rightarrow (\partial v/\partial x_1) = 0, \quad (\partial v/\partial x_2) = 0 \text{ on } \Gamma_0 \cap \Gamma_1 \Rightarrow b_1 = b_2 = 0 \\
&\Rightarrow v = b_0 \Rightarrow v = b_0 = 0, \qquad \text{since} \quad v|_{\Gamma_0} = 0 \\
&\Rightarrow V \cap P_1(\bar{\Omega}) = \{0\}.
\end{aligned}$$

Remark (5.2). In general, $V \equiv H_0^2(\Omega;\ \Gamma_0,\ \Gamma_1) \not\subset V_0$ defined in (5.10), since Γ_0 may not contain three noncollinear points a_1, a_2, a_3 and hence, Proposition (5.1) does not hold even when Proposition (5.2) holds.

COROLLARY (5.2). *Suppose that $\Gamma_0 = \Gamma_1 = \Gamma$ in* (2.4) *such that $V \equiv H_0^2(\Omega;\ \Gamma, \Gamma) \equiv H_0^2(\Omega)$* (2.8). *Then, assumption* (A4) *holds.*

6. Existence and Uniqueness of Solution of (P_G)

Theorem (6.1). *Under assumptions* (A2)–(A4), *the bilinear form* $a(\cdot,\cdot)$ *defined by* (4.4) *is V-elliptic; i.e.*, $\exists \alpha_1 > 0$ *such that*

$$a(v, v) \geqslant \alpha_1 \|v\|_V^2 \; \forall v \in V. \tag{6.1}$$

Proof. By virtue of (A3) and Theorem (5.1) on norm equivalence in V, we have

$$\forall v \in V, \; a(v, v) = \int_\Omega a_{ijkl} v_{,ij} v_{,kl} \, d\Omega \geqslant \alpha_0 \int_\Omega v_{,ij} v_{,ij} \, d\Omega$$

$$= \alpha_0 |v|_{2,\Omega}^2 \geqslant \alpha_1 \|v\|_{2,\Omega}^2 \quad \text{with} \quad \alpha_1 = \alpha_0 \beta_1 > 0.$$

Now, we can state the final result.

Theorem (6.2). *Under assumptions* (A1)–(A4), *the problem* (P_G) *has a unique solution.*

Proof. From Theorem (6.1), $a(\cdot,\cdot)$ is V-elliptic. The continuity of $a(\cdot,\cdot)$ follows from (A1) and the Cauchy–Schwarz inequality, and $|l(v)| \leqslant M \|v\|_V$ $\forall v \in V$, for some $M \geqslant 0$. Hence, the result follows from the Lax–Milgram lemma [3].

7. Examples

In [2] a good number of examples of practical interest—bending problems of clamped elastic anisotropic/orthotropic/isotropic plates and biharmonic (Stokes') problem of fluid mechanics—have been considered to show that all the assumptions (A1)–(A3) are satisfied by the coefficients a_{ijkl}, the explicit formulae for which are also given for each case in [2]. Hence, it remains to consider examples of different boundary conditions (i)–(iv) in (3.41) to show that the major assumption (A4), which the admissible space V of the corresponding Galerkin variational problem (P_G) must satisfy for the existence and uniqueness of solution of (P_G), holds.

For this we should consider only different boundary value problems of bending analysis of elastic anisotropic/orthotropic/isotropic plates with variable/constant thickness, which can be retrieved from (3.1), (3.2) with proper choices of coefficients a_{ijkl} satisfying (A1)–(A3) as shown in [2]. Therefore, for the proof of existence and uniqueness of solution in each case of a boundary value problem, it is sufficient to define the admissible space V explicitly and show that (A4) holds for this choice of V, the results being valid for anisotropic/orthotropic/isotropic cases with variable/constant thickness.

EXAMPLE (7.1). For $\Gamma_0 = \Gamma_2 = \Gamma$, $\Gamma_1 = \Gamma_3 = \phi$ in (3.41), which satisfy (3.7), *the simply supported* plate bending problem is obtained with $V \equiv H^2(\Omega) \cap H_0^1(\Omega)$ (2.6), for which (A4) holds by virtue of Corollary (5.1).

EXAMPLE (7.2). For $\Gamma_0 = \Gamma_1 = \Gamma$ and $\Gamma_2 = \Gamma_3 = \phi$ in (3.41) satisfying (3.7), we get the *clamped* plate bending problem or *Dirichlet* problem [2] with $V \equiv H_0^2(\Omega)$ for which (A4) follows from Corollary (5.2).

EXAMPLE (7.3). For $\Gamma_0 = \Gamma_1 \subsetneq \Gamma$, $\Gamma_2 = \Gamma_3 = \Gamma - \bar{\Gamma}_0$ with $\bar{\Gamma}_0 \cup \bar{\Gamma}_2 = \Gamma$, $\Gamma_0 \cap \Gamma_2 = \phi$ in (3.41) and (3.7), we obtain the bending problem of a plate (partially) *clamped on* Γ_0 and *free along* Γ_2 with $V \equiv H_0^2(\Omega; \Gamma_0)$ (2.7), for which (A4) follows from Proposition (5.2) (see Fig. 7.1).

EXAMPLE (7.4). For $\Gamma_0 \subsetneq \Gamma$, $\Gamma_1 = \phi$, $\Gamma_2 = \Gamma$, $\Gamma_3 = \Gamma - \bar{\Gamma}_0$ in (3.41) with Γ_0 containing three noncollinear points and $\text{meas}(\Gamma_0) \neq 0$, (3.7) holds with $\bar{\Gamma}_0 \cup \bar{\Gamma}_3 = \Gamma$, $\Gamma_0 \cap \Gamma_3 = \phi$ (Fig. 7.2). Thus, we obtain the bending problem of a plate *simply supported* on Γ_0 and *free along* Γ_3 with $V \equiv H^2(\Omega) \cap H_0^1(\Omega; \Gamma_0)$ (2.5) for which (A4) holds by virtue of Proposition (5.1), since Γ_0 contains three noncollinear points.

EXAMPLE (7.5). For $\Gamma_0 = \Gamma$, $\Gamma_1 \subsetneq \Gamma$, $\Gamma_2 = \Gamma - \bar{\Gamma}_1$, $\Gamma_3 = \phi$ with $\bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \Gamma$, $\Gamma_1 \cap \Gamma_2 = \phi$ in (3.41) and (3.7), we obtain the case of a plate *clamped on* Γ_1 and *simply supported on* Γ_2 (Fig. 7.3) with $V \equiv H_0^2(\Omega; \Gamma, \Gamma_1)$ (2.4), for which (A4) follows from Proposition (5.2).

EXAMPLE (7.6). For $\Gamma_0 = \{a_i\}_{i=1}^N$, $N \geqslant 3$, $\Gamma_1 = \phi$, $\Gamma_2 = \Gamma_3 = \Gamma - \{a_i\}_{i=1}^N$ with Γ_0 containing at least three noncollinear points a_1, a_2, a_3, $\bar{\Gamma}_0 \cup \bar{\Gamma}_3 = \Gamma$ in (3.41) and (3.7), we get the case of a plate *supported at the corner points* $\{a_i\}_{i=1}^N$, $N \geqslant 3$ with

$$V = \{v : v \in H^2(\Omega), v(a_i) = 0, 1 \leqslant i \leqslant N\}, \quad N \geqslant 3, \tag{7.1}$$

for which (A4) holds by virtue of Proposition (5.1).

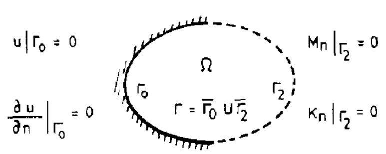

FIG. 7.1. Clamped on Γ_0, ////; free along Γ_2, ---.

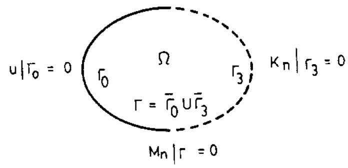

FIG. 7.2. Simply supported on Γ_0, —; free along Γ_3, ---.

Remark (7.1). By virtue of Sobolev's imbedding theorem [1, 3], $V \subsetneq C^0(\bar{\Omega}) \Rightarrow v|_\Gamma \in C^0(\Gamma)$.

EXAMPLE (7.7). For a *symmetric simply supported* plate problem (P) on $\bar{\Omega} = [-1, 1] \times [-1, 1]$ (see Example (7.1)), which is symmetric with respect to the x_1- and x_2-axes, we can define an auxiliary boundary value problem (P*) along with *symmetry conditions on a quadrant*, say the first quadrant $\bar{\Omega}^* = \Omega^* \cup \Gamma^* = [0, 1] \times [0, 1]$, as follows:

$$(P^*):\quad \Delta u = f \qquad \text{in} \quad \Omega^* = (0, 1) \times (0, 1);$$

$$\text{(i) } u|_{\Gamma_0^*} = 0, \quad \text{(ii) } \partial u/\partial n|_{\Gamma_1^*} = 0, \quad \text{(iii) } M_n|_{\Gamma_2^*} = 0, \quad \text{(iv) } K_n|_{\Gamma_3^*} = 0, \tag{7.2}$$

where

$$\Gamma_0^* = \Gamma_2^* = \{(x_1, x_2)\colon x_i = 1 \text{ with } x_j \in (0, 1),\ 1 \leqslant i \neq j \leqslant 2\},$$

$$\Gamma_1^* = \Gamma_3^* = \{(x_1, x_2)\colon x_i = 0 \text{ with } x_j \in (0, 1),\ 1 \leqslant i \neq j \leqslant 2\},$$

$$\bar{\Gamma}_0^* \cup \bar{\Gamma}_3^* = \bar{\Gamma}_1^* \cup \bar{\Gamma}_2^* = \Gamma^*, \qquad \Gamma_0^* \cap \Gamma_3^* = \Gamma_1^* \cap \Gamma_2^* = \phi,$$

(ii) and (iv) are symmetry conditions along $\Gamma_1^* = \Gamma_3^*$.

Thus, all the conditions (3.7) hold with Γ_i replaced by Γ_i^* ($i = 0, 1, 2, 3$) and $V = H_0^2(\Omega^*; \Gamma_0^*, \Gamma_1^*)$ for which (A4) holds by virtue of Proposition (5.1).

Remark (7.2). For other symmetric boundary value problems and also for other symmetric $\bar{\Omega}$, we can consider auxiliary boundary value problems (P*) as in (7.2).

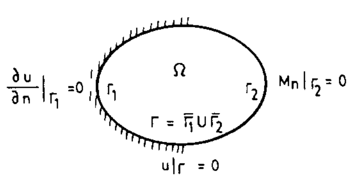

FIG. 7.3. Clamped on Γ_1, ⟋⟋⟋; simply supported on Γ_2, —.

Remark (7.3). Such a strategy of construction of auxiliary boundary value problems (P*) is applied in the finite element solution of (P) to reduce the volume of computations.

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